The complex sine-Gordon model on a half line

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Abstract: In this paper, we examine the complex sine-Gordon model in the presence of a boundary, and derive boundary conditions that preserve integrability. We present soliton and breather solutions, investigate the scattering of particles and solitons off the boundary and examine the existence of classical solutions corresponding to boundary bound states.

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Contents

1. Introduction

Two dimensional integrable field theories have become an area of extensive study. Their rich underlying mathematical structure allows for their exact solution. This in turn provides valuable information about the wide range of physical phenomena which integrable field theories can be used to model, and more generally about non-perturbative field theory. A number of models have been studied in the presence of a boundary. This has led to results for both the classical and quantum scattering of objects, like particles and solitons, off the boundary. In particular Toda models [1], the sine-Gordon model [2], and the sinh-Gordon model [3] amongst others have been studied on the half-line.

In this paper we study the classical two dimensional complex sine-Gordon (CSG) model in the presence of a boundary. The model in the bulk is described by the following Lagrangian

$$\mathcal{L}_{\text{CSG}} = \frac{1}{2} \frac{\partial u \bar{\partial} u^* + \bar{\partial} u \partial u^*}{1 - \xi u u^*} - 4\beta u u^* \,, \tag{1.1}$$

where $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$, $z = (t-x)/2$, $\bar{z} = (t+x)/2$. The field u is complex, β is a real coupling constant and the parameter ξ can be changed by rescaling u and will henceforth be set to one. The Lagrangian is invariant under global phase rotations of the field u and this leads to a conserved $\mathrm{U}(1)$ charge.

As such, the complex sine-Gordon model comprises an integrable generalisation of the sine-Gordon theory, with an internal $\mathrm{U}(1)$ degree of freedom. It was first introduced independently by Lund and Regge as a model of relativistic vortices in a superfluid [4, 6] and by Pohlmeyer in a dimensional reduction of a $\mathrm{O}(4)$ non-linear σ-model [5]. It belongs to a class of generalisations of the sine-Gordon theory, which in the literature is referred to as homogeneous sine-Gordon theories. The latter appear as $G/\mathrm{U}(1)$ gauged Wess-Zumino-Witten models perturbed by a potential. These generalisations describe integrable perturbations of $c > 1$ conformal field theories and have been studied in the bulk in [7, 8]. The CSG model appears as the simplest case where $G = \mathrm{SU}(2)$ and describes integrable perturbations of $\mathbf{Z}_k$ parafermions by the first thermal operator [12, 19]. The quantum case was also studied by Dorey and Hollowood in [18], and Maillet and de Vega in [16].

The CSG model has found many applications in different fields of physics, from general relativity [9], to the description of propagating optical pulses in a non-linear medium [10]. In the latter, the CSG theory was used as a generalisation of the pioneering work of McCall and Hahn [11], where the simple sine-Gordon (SG) was used as a field theory description. The CSG theory provides for a more realistic description of optical pulses incorporating effects like frequency detuning and modulation, while at the same time the theory can be extended beyond the description of two-level atom systems to multi-level atom systems.

In the next section of this paper some aspects of the theory in the bulk will be reviewed. Classical solutions, including particles, solitons and breathers will be presented for the model in the bulk clarifying previous treatments. Unlike the sine-Gordon model where the soliton solutions are topological in nature, the CSG model solitons which carry a Nöether $\mathrm{U}(1)$ charge, have no obvious topological charge associated with them. This follows from the trivial vacuum structure of the model for $\beta > 0$, although as we shall explain chargeless solitons do carry the topological charge of the sine-Gordon model in a subtle way. We also discuss the existence of breathers and find a class of charged breather solutions.

In section 3, we consider the effect of introducing a boundary and suitable boundary conditions are found which preserve the integrability of the model.In order to do this we explicitly construct low-spin conserved charges by abelianising the Lax pair description of the model, and ensure that these are conserved on the half-line by imposing boundary conditions. We find a natural generalisation of the boundary conditions that Ghoshal and Zamolodchikov [20] introduced for the sine-Gordon theory.

The final section in the main body of the paper deals with the scattering of particles and solitons. We find the corresponding solutions and use them to calculate the classical time-delay. We also find solutions corresponding to boundary bound states. The paper concludes with some general remarks about the results presented, as well as a few open questions that are related with some interesting properties of the model.

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2. Classical aspects of the CSG theory

In this section we shall review some important features of the CSG theory and establish our notation. We present the equation of motion, the relation of the model with the sine-Gordon theory and review the mathematical background in which the CSG model can be regarded as a perturbed gauged Wess-Zumino-Witten model. Moreover vacuum, soliton and multi-soliton solutions are written down in a compact form. Finally, we clarify the existing confusion concerning the existence of breathers within this model and present explicit breather solutions.

2.1 Definition of the model

The Lagrangian of the model was presented in (1.1). We assume without loss of generality $\xi = 1$ since the parameter can be dropped by rescaling the field variables. The equation of motion that follows is

$$\partial\bar{\partial}u + \frac{u^*\partial u\bar{\partial}u}{1 - uu^*} + 4\beta u(1 - uu^*) = 0 \ . \tag{2.1}$$

The relation between this model and sine-Gordon becomes obvious if we substitute

$$u = \sin\phi \ e^{2i\eta}, \tag{2.2}$$

with ϕ and η real fields. We use this notation here which gives a more standard connection with the sine-Gordon theory but differs from previous treatments by $\phi \to \frac{\pi}{2} - \phi$. This takes (1.1) to

$$\mathcal{L} = \partial\phi\bar{\partial}\phi + 4\tan^2\phi\,\partial\eta\bar{\partial}\eta - 4\beta\sin^2\phi \ . \tag{2.3}$$

This is essentially the form of the Lagrangian derived by Lund and Regge, and Pohlmeyer. By taking the field η to be constant the sine-Gordon Lagrangian emerges. As was demonstrated by Bakas [12] the theory can be reformulated in terms of a gauged WZW action. The corresponding action principle can be written as

$$S = S_{\text{gWZW}} + S_{\text{pot}} \ . \tag{2.4}$$

The action term S_{gWZW} is the well known gauged WZW action

$$S_{\text{WZW}} = -\frac{1}{4\pi}\int_\Sigma dzd\bar{z}\ \text{Tr}(g^{-1}\partial gg^{-1}\bar{\partial}g) - \frac{1}{12\pi}\int_B \text{Tr}(\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g})$$
$$+\frac{1}{2\pi}\int \text{Tr}(-W\bar{\partial}gg^{-1} + \bar{W}g^{-1}\partial g + Wg\bar{W}g^{-1} - W\bar{W}) \ . \tag{2.5}$$

This action is defined in a three-dimensional manifold B whose boundary is our compactified normal two-dimensional space Σ. The field g is an SU(2) group element and $\tilde{g}$ is the extension of g to the three dimensional manifold. The last term introduces gauge fields W and $\bar{W}$ which act as Lagrange multipliers. The S_{pot} term is

$$S_{\text{pot}} = \frac{\beta}{2\pi}\int \text{Tr}(g\sigma_3g^{-1}\sigma_3) \ . \tag{2.6}$$

This term breaks conformal invariance and thus gives rise to massive states. Varying the action yields the CSG equations of motion which can be expressed in a zero curvature form [13]

$$[\partial + (g^{-1}\partial g + g^{-1}Wg + i\beta\lambda\sigma_3)\,,\ \bar\partial + (\bar W - \frac{i}{\lambda}g^{-1}\sigma_3 g)\,] = 0\,. \tag{2.7}$$

From the variation of the gauge fields W and $\bar W$, two constraint equations arise

$$\begin{aligned}
\bar\partial g g^{-1} - g\bar W g^{-1} + \bar W &= 0\,,\\
g^{-1}\partial g + g^{-1}Wg - W &= 0\,,
\end{aligned} \tag{2.8}$$

which are critical in order to make the identification with the CSG theory. The connexion between the SU(2) matrix g and the complex field u of (2.1) is given by

$$g = \begin{pmatrix} u & -iv^* \\ -iv & u^* \end{pmatrix}\,, \tag{2.9}$$

where $v = -\sqrt{1 - uu^*}\,e^{-i\theta}$. The field variable θ should not be considered as an independent field but rather as an auxiliary field that is properly defined up to a constant through the constraint equations (2.8). In the gauge where $W = \bar W = 0$, the constraint equations take the form

$$\partial\theta = -i\frac{u^*\partial u - u\partial u^*}{2(1 - uu^*)}\ ,\quad \bar\partial\theta = -i\frac{u\bar\partial u^* - u^*\bar\partial u}{2(1 - uu^*)}\ , \tag{2.10}$$

whilst the equation of motion now becomes

$$[\partial - A, \bar\partial - \bar A] = 0\,, \tag{2.11}$$

where

$$A = -(g^{-1}\partial g + i\beta\lambda\sigma_3)\ ,\quad \bar A = \frac{i}{\lambda}g^{-1}\sigma_3 g\ . \tag{2.12}$$

This compact zero-curvature form of the equations of motion demonstrates the integrability of the model and will prove useful when we come to consider Bäcklund transformations and conserved quantities in later sections.

2.2 Vacuum solutions in the bulk

From (1.1) it is easy to see that the energy of the CSG model in the bulk is

$$\mathcal{H}_{\text{bulk}} = \int dx \left(\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - uu^*} + 4\beta uu^* \right)\ . \tag{2.13}$$

The most suitable candidate for a vacuum, would be a constant value for the field u that would force the kinetic term involving derivatives to vanish and at the same time minimize the potential term. It is clear to see from (2.2) and (2.3) that for $\beta > 0$ the obvious choice is $u = 0$, while for $\beta < 0$ the choice should be $|u| = 1$ if we insist that $|u| \leq 1$. The sign of the parameter β divides the theory into two sectors. In the matrix potential formalism, both sectors are treated simultaneously as the diagonal and off-diagonal parts of the field variable g. In this context the fields u and v are both solutions to the CSG equation each

derived for a specific choice of β, and each corresponding to a different vacuum . This is because the two sectors are connected by a duality transform which interchanges the sign of the coupling constant β [14] and simultaneously interchanges the role of u and v . That is to say the theory is invariant under the change

$$g \rightarrow g' = i\sigma_1 g = \begin{pmatrix} v & iu^* \\ iu & v^* \end{pmatrix} \;\; , \;\; \beta \rightarrow -\beta, \tag{2.14}$$

the latter representing a transform akin to the Krammers-Wannier duality of the Z_n parafermion theory [15]. Taking into account the invariance of the theory under this duality transform, we shall concentrate in this paper on the $\beta > 0$ sector which corresponds to the diagonal part of the matrix formalism. A suitable vacuum solution would be

$$g_{\text{vac}} = \begin{pmatrix} 0 & ie^{-i\Omega} \\ ie^{i\Omega} & 0 \end{pmatrix} \;. \tag{2.15}$$

This selection is consistent with the choices appearing in the beginning of this section with the diagonal $\beta > 0$ sector, corresponding to the $u = 0$ vacuum , while the off-diagonal $\beta < 0$ to $|v| = 1$. It is noted that the apparent singular behaviour of the Lagrangian at $|u| = 1$ does not appear as a problem embedded in the theory but is a direct consequence of the fact that the gauge fields $W, \bar{W}$ are ill defined at the specific point.

2.3 Spectrum of the model

The CSG model, like the sine-Gordon theory, possesses both particle and soliton solutions. When small perturbations around the vacuum are considered

$$u = 0 + \epsilon(x,t) , \tag{2.16}$$

the theory becomes linear when higher order terms in ϵ are ignored

$$(\partial_0^2 - \partial_1^2)\epsilon(x,t) + m^2\epsilon(x,t) = 0, \tag{2.17}$$

where $m^2 = 4\beta$. The solution to the above equation is the familiar plane waves solution

$$\epsilon(x,t) = e^{-i\omega t}\left(Ae^{ikx} + Be^{-ikx}\right), \tag{2.18}$$

where k and ω are related through

$$\omega^2 = k^2 + m^2. \tag{2.19}$$

Different techniques have been used for the construction of soliton solutions like the inverse scattering method [16] and the Hirota method [17]. However both methods yield results that are both cumbersome and difficult to manipulate. The Bäcklund transformation for the CSG model provides a more elegant way to obtain soliton solutions and can be written in terms of two matrix variables g and f [14]

$$g^{-1}\partial g - f^{-1}\partial f - \frac{\delta\beta}{\sqrt{|\beta|}}[g^{-1}\sigma_3 f, \sigma_3] = 0 , \tag{2.20}$$

$$\bar{\partial}gg^{-1}\sigma_3 - \sigma_3\bar{\partial}ff^{-1} + \frac{\sqrt{|\beta|}}{\delta}(gf^{-1}\sigma_3 - \sigma_3 gf^{-1}) = 0 . \tag{2.21}$$

It is easy to show that both f and g satisfy the CSG equation as well as the constraint equation in the specific gauge choice. Taking f to be an already known solution, one can generate a new solution through the equations presented above. One-soliton solutions can be derived by applying the Bäcklund transformation on the vacuum solutions g_{vac} of (2.15). Each sector of the theory provides us with two sets of two first order differential equations that can be integrated, in order to provide the one-soliton solutions. The diagonal elements of g which correspond to the $\beta > 0$ sector give

$$\partial_0 u - \sqrt{\beta} \left(\delta \, e^{i(\theta+\Omega)} - \frac{1}{\delta} \, e^{-i(\theta+\Omega)} \right) u\sqrt{1 - uu^*} = 0 \ ,$$

$$\partial_1 u + \sqrt{\beta} \left(\delta \, e^{i(\theta+\Omega)} + \frac{1}{\delta} \, e^{-i(\theta+\Omega)} \right) u\sqrt{1 - uu^*} = 0 \ . \tag{2.22}$$

The one-soliton solution that emerges is

$$u = \frac{\cos(a) \exp\left(2i\sqrt{\beta}\sin(a) \, \frac{t - Vx}{\sqrt{1-V^2}} \right)}{\cosh\left(2\sqrt{\beta}\cos(a)\frac{x-Vt}{\sqrt{1-V^2}} \right)} \ , \tag{2.23}$$

where V and a are real parameters associated with the velocity and charge of the soliton respectively. This solution was originally derived by Getmanov [17] for the $\beta > 0$ case. In addition an expression for the phase θ which appears in its dual field v, is also obtained

$$\theta = -\Omega - \arctan\left(\tan(a)\coth\left(2\sqrt{\beta}\cos(a)\frac{x-Vt}{\sqrt{1-V^2}} \right) \right) \ . \tag{2.24}$$

Respectively for the off-diagonal elements that correspond to the $\beta < 0$ sector, the set of equations is

$$\partial_0 v - \sqrt{|\beta|} \, e^{i\Omega} \left(\delta - \frac{1}{\delta} \right) (1 - vv^*) = 0 \tag{2.25}$$

$$\partial_1 v + \sqrt{|\beta|} \, e^{i\Omega} \left(\delta + \frac{1}{\delta} \right) (1 - vv^*) = 0 \, ,$$

that finally produce a different solution

$$v = -e^{i\Omega} \left(\cos(a)\tanh\left(2\sqrt{|\beta|}\cos(a)\frac{x-Vt}{\sqrt{1-V^2}} \right) + i\sin(a) \right) , \tag{2.26}$$

with Ω a real parameter associated with the vacuum of the theory. This is the solution that was derived by Lund and Regge [4] when considering the $\beta < 0$ case.

A two-soliton solution can be obtained through a non-linear superposition technique. Starting from the vacuum of the theory and by the application of the Bäcklund transformation twice, a set of parameters $\{\delta_1, \delta_2\}$ is used respectively in each step. The same procedure is followed again where the two parameters are used in the opposite order. By demanding that the two results are equal, one ends up with an equation for the two-soliton solution in matrix form

$$g_{2s} = \sigma_3 \left(\delta_1 g_2 - \delta_2 g_1 \right) g_{\text{vac}} \sigma_3 \left(\delta_1 g_1^{-1} - \delta_2 g_2^{-1} \right)^{-1} \ . \tag{2.27}$$

The matrix field variables g_k are of the general form of (2.9), with elements

$$u_k = \frac{\cos{(a_k)}N_k \exp(2i\sqrt{\beta}\sin{(a_k)}\Theta_k)}{\cosh\left(2\sqrt{\beta}\cos{(a_k)}\Sigma_k\right)} , \qquad (2.28)$$

$$v_k = -e^{i\Omega}\left(\cos{(a_k)}\tanh\left(2\sqrt{\beta}\cos{(a_k)}\Sigma_k\right) + i\sin{(a_k)}\right) . \qquad (2.29)$$

The identification one must make is:

$$\Sigma_k = \frac{1}{2}\left(\delta_k + \frac{1}{\delta_k}\right)x + \frac{1}{2}\left(\delta_k - \frac{1}{\delta_k}\right)t , \qquad (2.30)$$

$$\Theta_k = \frac{1}{2}\left(\delta_k + \frac{1}{\delta_k}\right)t + \frac{1}{2}\left(\delta_k - \frac{1}{\delta_k}\right)x , \qquad (2.31)$$

where N_k is a total phase. As expected g_{2s} has the same general form of equation (2.9)

$$g_{2s} = \begin{pmatrix} u_{2s} & -iv_{2s}^* \\ -iv_{2s} & u_{2s}^* \end{pmatrix} . \qquad (2.32)$$

The two-soliton solution and its complex conjugate are given by the diagonal elements of g_{2s}

$$u_{2s} = \frac{\left(-\delta_1 v_2^* + \delta_2 v_1^*\right)e^{i\Omega}\left(\delta_1 u_1 - \delta_2 u_2\right) + \left(-\delta_1 u_2 + \delta_2 u_1\right)e^{-i\Omega}\left(-\delta_1 v_1 + \delta_2 v_2\right)}{\delta_1{}^2 + \left(-u_1^* u_2 - u_2^* u_1 - v_1^* v_2 - v_2^* v_1\right)\delta_2\delta_1 + \delta_2{}^2} , \qquad (2.33)$$

$$u_{2s}^* = \frac{\left(-\delta_1 v_2 + \delta_2 v_1\right)e^{-i\Omega}\left(\delta_1 u_1^* - \delta_2 u_2^*\right) + \left(-\delta_1 u_2^* + \delta_2 u_1^*\right)e^{i\Omega}\left(-\delta_1 v_1^* + \delta_2 v_2^*\right)}{\delta_1{}^2 + \left(-u_1^* u_2 - u_2^* u_1 - v_1^* v_2 - v_2^* v_1\right)\delta_2\delta_1 + \delta_2{}^2} , \qquad (2.34)$$

while the off diagonal elements represent the dual field and its conjugate

$$v_{2s} = \frac{\left(-\delta_1 u_2^* + \delta_2 u_1^*\right)e^{i\Omega}\left(\delta_1 u_1 - \delta_2 u_2\right) - \left(-\delta_1 v_2 + \delta_2 v_1\right)e^{-i\Omega}\left(-\delta_1 v_1 + \delta_2 v_2\right)}{\delta_1{}^2 + \left(-u_1^* u_2 - u_2^* u_1 - v_1^* v_2 - v_2^* v_1\right)\delta_2\delta_1 + \delta_2{}^2} , \qquad (2.35)$$

$$v_{2s}^* = \frac{\left(-\delta_1 u_2 + \delta_2 u_1\right)e^{-i\Omega}\left(\delta_1 u_1^* - \delta_2 u_2^*\right) - \left(-\delta_1 v_2^* + \delta_2 v_1^*\right)e^{i\Omega}\left(-\delta_1 v_1^* + \delta_2 v_2^*\right)}{\delta_1{}^2 + \left(-u_1^* u_2 - u_2^* u_1 - v_1^* v_2 - v_2^* v_1\right)\delta_2\delta_1 + \delta_2{}^2} . \qquad (2.36)$$

The expressions above represent two-soliton solutions to the equation of motion and are related through the duality transformation of (2.14).

Multi-soliton solutions can also be obtained by following the same technique. Instead of the vacuum solution, one can start from any given n-soliton solution S_n and add solitons through the method described above, ending up with a S_{n+2} solution. Nevertheless, multi-soliton solutions for this model are quite large and their calculation is beyond the scope of this paper.

2.4 Soliton - Antisoliton duality

In this section we argue that soliton-soliton and soliton-antisoliton solutions presented in the literature by previous treatments, do not represent distinct classes of solutions. Charged solitons are non-topological solutions, therefore a distinction between a soliton
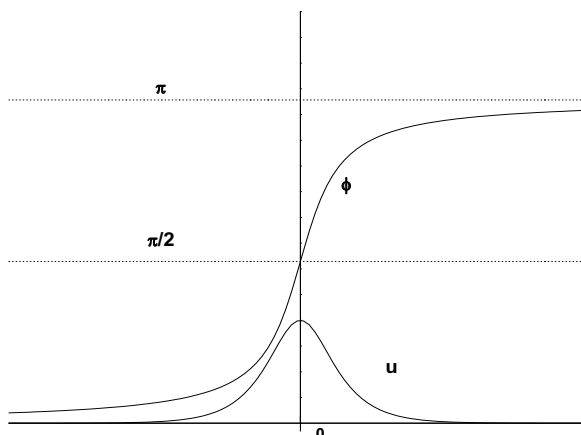
Figure 1: The ϕ and u solitons.

and an antisoliton is impossible. On the other hand chargeless solutions may be realised as topological solitons and identified with the sine-Gordon solitons.

The sine-Gordon theory appears as the limit of the CSG model when the charge parameter a is set to zero. We can substitute in the equation of motion of (2.1)

$$u = \sin \phi e^{2i\eta} \ , \tag{2.37}$$

where η is now a constant to recover the sine-Gordon model in the usual form

$$\partial_0^2 \phi - \partial_1^2 \phi + 2\beta \sin 2\phi = 0 \ . \tag{2.38}$$

The sine-Gordon theory has topological solitons (both kinks and antikinks) interpolating between its degenerate vacuaa. In contrast the CSG theory has a single vacuum for $\beta > 0$, and therefore its solitons are not topological in nature, but are stable because of integrability alone. The topological nature is hidden within the mapping of (2.37) and one has to be careful when trying to recover the sine-Gordon soliton as a limit of the CSG theory. Nonetheless a subtle remnant of the topology survives the mapping to the complex sine-Gordon theory. To see this consider how a SG soliton is mapped to CSG soliton. This is shown in figure 1.

Consider now how the potential term behaves as x increases for the single soliton solution. As a function of u we can express the potential as

$$\sin 2\phi = 2 \sin \phi \cos \phi = \pm 2u\sqrt{1 - uu^*} \ , \tag{2.39}$$

where η has been ignored as a total phase. Note that we should take the branch cut with opposite signs on each side of the point $\phi = \frac{\pi}{2}$, $u = 1$. We shall see that the changing sign

of the branch cut for a chargeless soliton will be important when we come to consider the theory with a boundary.

In some sense the topology of the chargeless u-soliton is embedded in the the branch cut that appears at the singular point $u = 1$. The choice of branch corresponds to a different vacuum for ϕ and therefore to a different topological charge.

However, when the charge parameter a is not zero, then the u-soliton does not reach the sick point $u = 1$ and remains non-topological. In this case no real distinction can be made between a soliton and an antisoliton. In the sine-Gordon case, the antikink solution is derived from the kink by changing the sign of the parameter δ of the Bäcklund transformation. This effectively corresponds to a parity and time reversal transformation which finally produces an antikink solution. Examining (2.28) we see that in the CSG case this change actually leads to the complex conjugate solution, by changing the sign of the complex phase. The change of sign in both t and x, can be cancelled by taking the charge parameter $a \rightarrow -a$. It is thus clear that instead of changing the sign of δ, one could effectively change the sign of a to derive an antisoliton. Since the soliton solution is a smooth function in a, the antisoliton is not a distinct object but can be identified with the soliton itself.

This also has an effect on the two-soliton solution. If we follow the same steps as in the sine-Gordon two-soliton solution then the solution u_{2s} of (2.33) corresponds to both a soliton-soliton and a soliton-antisoliton solution depending on the choice of sign for the Bäcklund parameter δ_2. The parameter δ_2 can be chosen in such a way as to describe one of the following

$$\delta_1 = -(\delta_2)^{-1} = \sqrt{\frac{1-V}{1+V}} \qquad \text{soliton-soliton scattering} \qquad (2.40)$$

$$\delta_1 = (\delta_2)^{-1} = \sqrt{\frac{1-V}{1+V}} \qquad \text{soliton-antisoliton scattering} \qquad (2.41)$$

Here we have taken the two solitons to have equal and opposite velocity (In general this is not the centre of mass since differently charged solitons have different masses, but it will be convenient for our discussion when we introduce a boundary later on). Also for reasons of simplicity we will refer to the soliton-soliton solution as u_{ss} ($\delta_2 = -1/\delta_1$) and to the soliton-antisoliton as u_{sa} ($\delta_2 = 1/\delta_1$).

However since no topological distinction exists between the soliton and antisoliton sector, it is possible to find a transformation of the parameters of the solution which effectively acts as a change of sign for the parameter δ_2. In fact, a set of transformations exists that maps u_{ss} to u_{sa} but we restrict ourselves to the simplest cases.

Before introducing the transformation, we need to introduce arbitrary shifts in x, which are crucial not only for this mapping but also later when we consider breathers and soliton reflections. The shifts appear in exponentials, so it is more helpful to consider the shifts in the following forms

$$K_i = \exp\left(2\sqrt{\beta}\cos a_i \frac{x_i}{\sqrt{1-V^2}}\right)$$

$$J_i = \exp\left(2i\sqrt{\beta}\sin a_i \frac{V y_i}{\sqrt{1-V^2}} + iR_i\right); \quad i = 1, 2 . \tag{2.42}$$

The parameters K_i, J_i are directly related with both V and a and correspond to the arbitrary initial positions in x, in the real (Σ_i) and imaginary phases (Θ_i) respectively, that appear in the one-soliton solution. Specifically the parameter K represents a translation in x, while the J parameter represents a phase shift in the internal $\text{U}(1)$ space. For reasons of simplicity, we include in the definition of J the total phase $N_k = \exp(iR_k)$ which appears in (2.28). Henceforth these parameters will be referred as phase shifts, since they are directly related to the time-delay effect of the scattering process.

Now that we have defined the arbitrary phase shifts we start with the soliton-soliton solution u_{ss} which comes from the two-soliton solution u_{2s} when we choose $\delta_2 = -1/\delta_1$. We consider the following transformation

$$a_2 \rightarrow -a_2 . \tag{2.43}$$

Although this is enough to change a single soliton to an antisoliton, this is not the case for the two-soliton solution. The phase shifts have also to be fixed in a specific way to complete the mapping between u_{ss} and u_{sa}

$$J_1 \rightarrow -J_1 \tag{2.44}$$

$$K_2 \rightarrow 1/K_2 . \tag{2.45}$$

This effectively changes the sign of δ_2 in the expression u_{ss} converting one of the solitons to an antisoliton. In contrast with the single soliton where the antisoliton can not be properly defined, in the two-soliton case there is a point of reference. A distinction between a soliton and an antisoliton can only be realised as a specific choice of the relative sign between the parameters a_1, a_2 and V which does not in any case lead to topologically distinct solutions.

The same mapping between u_{ss} and u_{sa} can also be achieved by making the following transformation

$$a_2 \rightarrow a_2 + \pi , \tag{2.46}$$

which effectively changes the sign of all trigonometric functions involving the parameter a_2 sending the solution $u_{\text{ss}} \rightarrow -u_{\text{sa}}$. This transformation will be used again on a later section when we come to consider soliton reflections, to demonstrate exactly the equivalence of the two sets of solutions.

2.5 Breather solutions

There are conflicting views in the literature concerning the existence of breathers [17, 18]. The problem arises because the transformation $V \rightarrow iV$ which is usually used to generate breathers from a two-soliton solution traveling with equal and opposite velocities, does not necessarily lead to a solution of the equations of motion. While the technique has been widely used before on other models, the fact that the CSG equation involves both u and u^*, implies that naively analytically continued solutions do not necessarily satisfy the equation of motion.

So it is not clear, for instance, that all the breather like "solutions" of [14] do satisfy the CSG equations of motion. However, since the sine-Gordon is embedded in CSG model by taking u to be chargeless, the sine-Gordon breather solutions do satisfy the CSG equations of motion. In fact a family of charged, complex breather solutions does exist in CSG model. Although it is quite hard to actually check if a general breather solution satisfies the equation of motion, a trick can be used instead. We consider the two-soliton solution of (2.33) and we demand that this solution is even in V so that is effectively a function of V^2. Now the transformation $V \to iV$, doesn't change the reality properties of the solution but simply introduces an overall minus sign into the arbitrary parameter V^2, which is irrelevant. Making the solution even in V, means that a few restrictions have to be imposed. Firstly, the charge parameters have to be taken equal or opposite according whether $\delta_1 \delta_2$ is plus or minus one respectively. Secondly, some of the arbitrary position parameters, have now to be fixed. However, up until now all the arbitrary phase shifts that appeared were either complex (J_i) or real (K_i) and there was no distinction between the shifts that originated from the space or time part of the phase. However, when constructing a breather solution, by analytical continuation of the V parameter a separation between the space and time shifts is induced. All shifts that associated with space end up as real parameters, while time shifts become imaginary. We can restrict ourselves to shifts only in the x direction. One could also consider more general phases which are complex and also depend on time and the parameter V. These however correspond to either U(1) rotations or time translations which make their use obsolete. The arbitrary shift parameters are now both real

$$K_s = \exp\left(2\sqrt{\beta}\cos a_s \frac{x_s}{\sqrt{1+V^2}}\right)$$
$$J_s = \exp\left(2\sqrt{\beta}\sin a_s \frac{V y_s}{\sqrt{1+V^2}}\right) \; ; \quad s = 1, 2 \; . \tag{2.47}$$

and should be compared with the general form of (2.42). In order to make a breather solution from the soliton-soliton case the following relations are required

$$K_1 = \pm\frac{1}{K_2} \text{ and } J_1 = \mp\frac{1}{J_2} \; , \tag{2.48}$$

where the signs in these equation are correlated.

It should be noted that more breather solutions may exist. It is possible that through certain restrictions a more general breather solution can be obtained, but a direct confirmation through the equations of motion is rather difficult.

2.6 Collapse of a Breather

An analysis of the quantum CSG model [18] suggests that the soliton can be identified with the elementary particle since the vacuum of the theory and the one-soliton are not topologically distinct solutions. Evidence of this conjecture exists even in the classical picture. From our experience with the sine-Gordon model, we would expect to identify the particle with the lowest energy breather solution. It would seem to follow that the breather

whose energy and charge correspond to that of a single particle should be equivalent to a single soliton. This remarkable fact can be shown as follows.

We consider the static single-soliton solution

$$u_s = \frac{\cos(a)\exp(im\sin(a)t)}{\cosh(m\cos(a)(x+x_0))} \ , \tag{2.49}$$

where $m = 2\sqrt{\beta}$. The mass of the static soliton is given by (2.13) which after integration gives

$$M_s = 4m\cos(a_s) \,, \tag{2.50}$$

while the charge of the soliton is given by

$$Q_s = i\int \frac{u^*\partial_0 u - u\partial_0 u^*}{1-uu^*} = 4\left(\text{sign}[a_s]\frac{\pi}{2} - a_s\right) \ . \tag{2.51}$$

The mass of a breather is twice the mass of a single soliton solution at velocity V, which has been analytically continued $V \to iV$

$$M_B = \frac{8m\cos(a_B)}{\sqrt{1+V_B{}^2}} \ . \tag{2.52}$$

The breather solution is effectively constructed from two one-soliton solutions, each with charge

$$Q_B = 4\left(\text{sign}[a_B]\frac{\pi}{2} - a_B\right) \ . \tag{2.53}$$

In order to have a chance of identifying the breather with the soliton, we demand that the mass of a breather is equal to the mass of a static single soliton and that their charges also coincide

$$M_s = M_B \ , \ \ Q_s = 2Q_B \ . \tag{2.54}$$

From the above relations, one can solve for the parameter V_B

$$V_B = \sqrt{\frac{2\cos(a_B)}{\cos(a_s)} - 1} \ . \tag{2.55}$$

If this value is substituted into the breather, then the solution collapses to a static single-soliton carrying double the charge Q_B. In other words, the single-soliton can always be considered as a bound state of two single-solitons carrying half the charge. The argument can be used recursively so that a soliton can be regarded as an infinite collection of solitons carrying fractions of the original charge. At each level a soliton is identified with a breather emerging out of a soliton pair of half the original charge. In the classical picture this process can be carried out indefinitely, but in the quantum case the finite character of the mass states restricts this procedure.

This is not surprising since the static single-soliton of (2.49) can be viewed as a bound state due to the oscillation effect which creates a breather-like behaviour. This is consistent with the fact that any breather can collapse to to this solution when the parameter V is properly fixed. It can therefore be realised as a breather solution after the collapse, exhibiting all of its former properties.

One point that has to be emphasized is that breathers constructed with the method described in the previous section are not chargeless. This is due to the fact that the choice of the charge parameters a_i is such that both solitons that are combined to create a breather have the same charge. This is confirmed by the above demonstration in which a breather collapses to a single soliton solution which carries double the charge of the breather's solitons. Neutral breathers do exist but only at the chargeless limit and can be identified with the breathers of the simple sine-Gordon theory.

3. Reflections

In the following sections we consider the effect of introducing a boundary into the CSG model. Although we are free to add any boundary potential to the Lagrangian, we choose to investigate those potentials and their corresponding boundary conditions which preserve integrability. Such choices allow the non-perturbative solution of the model which is one of the main motivation for its study. Once such suitable conditions are introduced we examine the scattering of particles and solitons, and determine the necessary conditions for the existence of boundary bound states.

3.1 Abelianisation of the Lax pair and conserved currents

We shall consider a boundary condition to have preserved the integrability of the CSG model, if we can still construct an infinite number of commuting conserved charges. In contrast with the theory in the bulk, the introduction of a boundary destroys the translation invariance of the model but preserves the time translation invariance. It is thus expected that the momentum will not be conserved, whilst the energy will. This situation also holds for the higher-spin conserved quantities. All energy-like, parity-even quantities can be conserved, unlike their momentum-like, parity-odd partners. Nevertheless, since there is an infinite number of conservation laws, the main goal would be to concentrate on the conservation of the parity-even quantities.

The presence of the spectral parameter λ in the Lax pair of (2.11) implies the existence the infinite conserved currents in the bulk that can be determined through the method used by Turok and Olive [21]. This is achieved by performing a gauge transformation U

$$\mathcal{A} = U A U^{-1} + \partial U U^{-1} \,, \tag{3.1}$$

in such a way that the commutator of the transformed gauge fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ of the Lax pair to be zero. The equation of motion becomes

$$\partial_0(\bar{\mathcal{A}} - \mathcal{A}) = \partial_1(\bar{\mathcal{A}} + \mathcal{A}) \,, \tag{3.2}$$

where the normal time and space derivatives are used. In the theory in the bulk we integrate over x. If $Q(\lambda) = \int_{\infty}^{+\infty}(\bar{\mathcal{A}} - \mathcal{A})dx$, then

$$\frac{d}{dt}Q(\lambda) = \int_{-\infty}^{\infty} \partial_0(\bar{\mathcal{A}} - \mathcal{A}) \, dx = \int_{-\infty}^{\infty} \partial_1(\bar{\mathcal{A}} + \mathcal{A}) \, dx = \left[(\bar{\mathcal{A}} + \mathcal{A})\right]_{-\infty}^{\infty} \,. \tag{3.3}$$

Since at infinity the fields are taken to vanish so that $\mathcal{A}$ and $\bar{\mathcal{A}}$ approach a fixed value, it follows that the right-hand side vanishes. As $\mathcal{A}$ and $\bar{\mathcal{A}}$ can be expanded as an infinite Laurent series in λ, the coefficients of each power of λ, provides us with an infinite number of conserved charges.

When a boundary is introduced the left-hand side involving the spatial derivative does not vanish since now the integration takes place over the semi-infinite interval. Instead one is left with an equation of the form

$$\int_{-\infty}^{0} \partial_0(\bar{A} - A) \, dx = \left[(\bar{A} + A)\right]_{x=0} \tag{3.4}$$

where the left-hand side is evaluated at the boundary. Instead of demanding that the right-hand side vanishes, we instead ask that it can be expressed as a total time derivative with the help of suitable conditions, thus leading to a conserved quantity.

We begin by finding explicit expressions for "low-spin" conserved charges of the CSG model in the bulk by solving for the abelianizing gauge transformation U order by order in the spectral parameter.

Let U be a general real SU(2) matrix, with $\det(U) = 1$. The diagonal elements of U can be taken equal due to residual gauge freedom which leave $\mathcal{A}$ and $\bar{\mathcal{A}}$ in an abelian form. Thus U takes the form

$$U = \frac{1}{\sqrt{1 - \chi\bar{\chi}}} \begin{pmatrix} 1 & \chi \\ \bar{\chi} & 1 \end{pmatrix} \ , \tag{3.5}$$

where χ is a function of the fields and should not be associated with the space variable. We demand that U diagonalises both A and $\bar{A}$ at the same time. The transformed fields lie both in the σ_3 direction and the non-zero diagonal elements can be identified with the conserved currents. Taking A to be

$$A = \begin{pmatrix} i\Lambda & E \\ -E^* & -i\Lambda \end{pmatrix}, \tag{3.6}$$

with $\Lambda = \beta\lambda$ and $E = i(u^*\partial v^* - v^*\partial u^*)$, we demand that the non-diagonal part of α vanishes

$$2i\Lambda\chi + \chi^2 E^* + E + \partial\chi = 0 \ ,$$

$$2i\Lambda\bar{\chi} + \bar{\chi}^2 E^* + E + \partial\bar{\chi} = 0 \ . \tag{3.7}$$

The conserved quantities can also be written in terms of χ and $\bar{\chi}$

$$J = -i\Lambda\frac{1 - \chi\bar{\chi}}{1 - \chi\bar{\chi}} - \frac{\chi E^* + \bar{\chi}E}{1 - \chi\bar{\chi}} + \frac{\chi\partial\bar{\chi} - \bar{\chi}\partial\chi}{2(1 - \chi\bar{\chi})} \ . \tag{3.8}$$

The same matrix U, should also diagonalise $\bar{A}$, which is given by (2.12)

$$\bar{A} = \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \ , \tag{3.9}$$

where $D = uu^* - vv^*$ and $P = -2iu^*v^*$. The choice of E, P and D is not accidental. They actually represent the electric field, the polarization and the population inversion field variables respectively, when this theory is used to describe the propagation of optical pulses in a non-linear medium [10]. When U acts on $\bar{A}$, we again demand the off diagonal parts to vanish. Examining the matrix explicitly yields

$$\frac{i}{\lambda}(-2D\chi + P - P^*\chi^2) + \bar{\partial}\chi = 0\,,$$

$$\frac{i}{\lambda}(2D\bar{\chi} + P^* - P\bar{\chi}^2) + \bar{\partial}\bar{\chi} = 0\,. \tag{3.10}$$

It is easy to see that these equations are equivalent to equations (3.7).

The diagonal part yields the other component of the conserved current

$$\bar{J} = \frac{i}{2\lambda}\left(2\frac{1+\chi\bar{\chi}}{1-\chi\bar{\chi}}D + 2\frac{1P^*\chi - \bar{\chi}P}{1-\chi\bar{\chi}}\right) + \frac{(\chi\partial\bar{\chi} - \bar{\chi}\bar{\partial}\chi)}{2(1-\chi\bar{\chi})}. \tag{3.11}$$

In order to solve the two sets of equations (3.7) or equivalently (3.10), we consider an expansion of χ and $\bar{\chi}$ in powers of Λ

$$\chi = \frac{\chi_1}{\Lambda} + \frac{\chi_2}{\Lambda^2} + \frac{\chi_3}{\Lambda^3} + \cdots\,,$$

$$\bar{\chi} = \frac{\bar{\chi}_1}{\Lambda} + \frac{\bar{\chi}_2}{\Lambda^2} + \frac{\bar{\chi}_3}{\Lambda^3} + \cdots\,. \tag{3.12}$$

The coefficients χ_i and $\bar{\chi}_i$ can be determined by direct substitution into (3.7) and (3.10), and by demanding that the coefficients in all powers of Λ vanish. Up to order $\mathrm{O}(\Lambda^{-2})$ one finds

$$\chi = \left(\tfrac{i}{2\Lambda}\right)E + \left(\tfrac{i}{2\Lambda}\right)^2 \partial E + \left(\tfrac{i}{2\Lambda}\right)^3 (E^2E^* + \partial^2 E) + \cdots\,,$$
$$\bar{\chi} = \left(\tfrac{i}{2\Lambda}\right)E^* + \left(\tfrac{i}{2\Lambda}\right)^2 \partial E^* + \left(\tfrac{i}{2\Lambda}\right)^3 (EE^{*2} + \partial^2 E^*) + \cdots\,, \tag{3.13}$$

Now that χ and $\bar{\chi}$ have been defined, we can also express the conserved quantities as a series in λ. Each order of λ, provides a conserved quantity and since the series of λ in χ and $\bar{\chi}$ does not terminate, we thus have an infinite number of conserved quantities as expected from the integrability of the CSG model. The two components of the conserved current up to $\mathcal{O}(\lambda^{-2})$ can be read off as coefficients in the following expansion of J and $\bar{J}$

$$J = -\lambda\beta - \frac{i}{2\beta}EE^*\left(\frac{1}{\lambda}\right) - \frac{1}{8\beta^2}(E\partial E^* - E^*\partial E)\left(\frac{1}{\lambda^2}\right) + \cdots\,, \tag{3.14}$$

$$\bar{J} = iD\left(\frac{1}{\lambda}\right) + \frac{1}{4\beta}(E^*P - EP^*)\left(\frac{1}{\lambda^2}\right) + \cdots\,, \tag{3.15}$$

and it can be checked that this current is conserved explicitly from the equation of motion.

In the above we have constructed conserved currents that lead to conserved charges in the bulk. However, as we have previously argued, conserved charges on the half line are expected to take the form of an integral over a *parity-even* conserved current. The conserved

currents above are neither parity even or odd. To rectify this we note that our system of equations and constraints possess a Z_2 invariance involving parity transformations which can be used to construct a "reflected" set of conserved currents. The "reflected" set of conserved currents is easily obtained through the substitution $\partial \to \bar{\partial}$ in the expressions (3.14) including those derivatives involved in the definition of E. The new set of currents $\widetilde{J}$, $\widetilde{\bar{J}}$, can now be combined with the former set to produce pure parity odd and even currents.

In the presence of a boundary only parity even quantities are conserved. The desired form of the equations to emerge is

$$\partial_0 \ (parity \ \ even) = \partial_1 \ (parity \ \ odd). \tag{3.16}$$

By combining the two sets of currents one can separate the odd and even quantities for all powers of λ.

$$\partial_0 \left[(\bar{J} + \widetilde{\bar{J}}) - (J + \widetilde{J}) \right] = \partial_1 \left[(J - \widetilde{J}) + (\bar{J} - \widetilde{\bar{J}}) \right] \tag{3.17}$$

We examine the λ^{-1} term in the expansion which gives

$$\partial_0 \left(EE^* + \tilde{E}\tilde{E}^* + 2\beta(D + \tilde{D}) \right) = \partial_1 \left(\tilde{E}\tilde{E}^* - EE^* + 2\beta(D - \tilde{D}) \right), \tag{3.18}$$

where $\tilde{E} = E(\partial \to \bar{\partial})$, etc. After integration over the semi-infinite interval, the right hand side representing the parity odd part is

$$\partial_0 \mathcal{W}(u, u^*) = \left(\frac{2\partial_1 u^*}{1 - uu^*} \right) \partial_0 u + \left(\frac{2\partial_1 u}{1 - uu^*} \right) \partial_0 u^* \ . \tag{3.19}$$

This is a total derivative provided that

$$\frac{2\partial_1 u^*}{1 - uu^*} = \frac{\partial \mathcal{W}}{\partial u} \ , \quad \frac{2\partial_1 u}{1 - uu^*} = \frac{\partial \mathcal{W}}{\partial u^*} \ . \tag{3.20}$$

The conserved quantity at hand, in terms of u and u^*, is then

$$\mathcal{H} = \int_{-\infty}^0 \left(2\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - |u|^2} + 4\beta(2|u|^2 - 1) \right) dx - [\mathcal{W}]_{x=0} \ , \tag{3.21}$$

Since this quantity actually represents the energy of the system, $\mathcal{W}$ can be identified with the energy contribution of the boundary term.

When constructing the odd and even quantities of the λ^{-2} term, one ends up with

$$\partial_0 \left(\frac{1}{2}(E^*\partial E - E\partial E^* + \tilde{E}\bar{\partial}\tilde{E}^* - \tilde{E}^*\bar{\partial}\tilde{E}) - \beta(E^*P - EP^* + \tilde{E}^*\tilde{P} - \tilde{E}\tilde{P}^*) \right) = \tag{3.22}$$

$$= \partial_1 \left(\frac{1}{2}(-E^*\partial E + E\partial E^* + \tilde{E}\bar{\partial}\tilde{E}^* - \tilde{E}^*\bar{\partial}\tilde{E}) - \beta(E^*P - EP^* - \tilde{E}^*\tilde{P} + \tilde{E}\tilde{P}^*) \right)$$

Once more the parity-odd right hand side which after integration yields

$$4\frac{\partial_0 u \partial_0 \partial_1 u^*}{1 - uu^*} - 4\frac{\partial_0 u^* \partial_0 \partial_1 u}{1 - uu^*} - 4\frac{(\partial_1 u \partial_1 u^* + \partial_0 u \partial_0 u^*)(u\partial_1 u^* - u^*\partial_1 u)}{(1 - uu^*)^2}$$

$$-4\beta(u\partial_1 u^* - u^*\partial_1 u) + 4\frac{\partial_1 u \partial_0^2 u^*}{1 - uu^*} - 4\frac{\partial_1 u^* \partial_0^2 u}{1 - uu^*} \ , \tag{3.23}$$

should be written as a total time derivative in order to force the currents to be conserved at the boundary. A set of boundary conditions have to be introduced to ensure that his is the case (appendix A). Using the equations of motion, the parity-odd part of (3.23) can be written as a total derivative if the following restrictions are enforced

$$\partial_1 u = -Cu\sqrt{1 - uu^*},$$
$$\partial_1 u^* = -Cu^*\sqrt{1 - uu^*}. \tag{3.24}$$

The boundary constant C is defined by the theory and is responsible for the way fields react to the boundary. Consistency of the two equations in (3.24) implies that C should be considered a real parameter. When one makes the transformation described in (2.2), the new boundary conditions for the fields ϕ and η are

$$\partial_1 \phi = -C\sin(\phi) \quad , \quad \partial_1 \eta = 0 \ , \tag{3.25}$$

which clearly shows that C has to be real.

It has to be pointed out that (3.24) is not the only set of boundary conditions that can be derived. A number of isolated "Dirichlet"-like conditions also exists. However, we restrict ourselves only to cases where the space derivatives of the fields appear. If we take the field u to be real, the system is reduced to the sine-Gordon equation with a boundary condition $\partial_1 \phi = -C\sin\phi$. This is the subset of integrable boundary conditions of the sine-Gordon model presenting the Z_2 symmetry $\phi \to -\phi$. The corresponding conserved quantity, is rather large and is omitted.

4. Soliton scattering and boundary bound states

Since the necessary conditions for the integrability of the model have been established, we study the scattering of particles and solitons off the boundary. We begin this section with the effects of introducing a boundary potential to the vacuum of the theory. We continue with the scattering of particles and solitons and derive the phase shifts induced by the process. Finally, we investigate the necessary conditions for the existence of boundary bound states.

4.1 Vacuum

When a boundary term is introduced, the vacuum of the theory that we discussed in section (2.2), does not necessarily remain unchanged. It is exactly this contribution that needs to be carefully examined before any statements are made about the minimum energy configuration. Although, the vacuum solution of the theory in the bulk is a strong candidate, soliton solutions could also be considered in the attempt to both minimize the energy functional and satisfy the boundary conditions of (3.24).

We begin by first determining the energy contribution of the boundary term. The full Lagrangian of the model is now

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\mathcal{B}}. \tag{4.1}$$

The boundary term $\mathcal{L}_\mathcal{B}$, can be determined by the variation principle of the total action. The variation of the $\mathcal{L}$ term yields

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial u}\delta u + \frac{\partial\mathcal{L}}{\partial u^*}\delta u^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u)}\delta(\partial_\mu u) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u^*)}\delta(\partial_\mu u^*) \ . \tag{4.2}$$

When the Euler-Lagrange equations are used, two terms survive since the model is considered in the semi-infinite interval where the fields do not vanish at the boundary

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu u)}\delta u\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu u^*)}\delta u^*\right) \ . \tag{4.3}$$

From the variation of the boundary term one has

$$\delta\mathcal{L}_\mathcal{B} = \frac{\partial\mathcal{L}_\mathcal{B}}{\partial u}\delta u + \frac{\partial\mathcal{L}_\mathcal{B}}{\partial u^*}\delta u^* \ . \tag{4.4}$$

The variation of the action vanishes when the remaining terms evaluated at the boundary are forced to cancel. The two interrelated equations that emerge are

$$\frac{\partial\mathcal{L}}{\partial(\partial_1 u^*)} = \frac{-\partial_1 u}{1 - uu^*} = -\frac{\partial\mathcal{L}_\mathcal{B}}{\partial u^*} \ , \tag{4.5}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_1 u^*)} = \frac{-\partial_1 u^*}{1 - uu^*} = -\frac{\partial\mathcal{L}_\mathcal{B}}{\partial u^*} \ . \tag{4.6}$$

By substituting the boundary conditions of (3.24), these can easily be solved for the boundary term

$$\mathcal{L}_B = 2C\sqrt{1 - uu^*} \ . \tag{4.7}$$

We now consider the total energy of the system , now comprising of two parts

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{bulk}} + \mathcal{H}_B \ , \tag{4.8}$$

where the term $\mathcal{H}_{\text{bulk}}$, represents the energy in the bulk and the second term $\mathcal{H}_B$ represents the energy contribution from the boundary

$$\mathcal{H}_B = -2C\sqrt{1 - uu^*} \ , \tag{4.9}$$

which is evaluated at $x = 0$. This energy contribution makes the determination of the vacuum difficult. The sign of the boundary constant C is not set, which could provide either a positive or negative contribution to the total energy of the system. This clearly shows that although the original choice for a vacuum should not be discarded, one should also consider other static solutions which in conjunction with the sign of C could provide a lower energy vacuum than before.

Apart from the original choice for a vacuum, one can consider static multi-soliton solutions. We restrict ourselves to one-soliton solutions since experience with similar models usually makes multi-soliton solutions unsuitable candidates.

When considering one-soliton solutions, one has the equations of the Bäcklund transformation (2.22) which are always true to simplify expressions. In particular we first consider the $\mathcal{H}_{\text{bulk}}$ term representing the energy in the bulk

$$\mathcal{H}_{\text{bulk}} = \int dx \left(\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - uu^*} + m^2 uu^* \right) , \tag{4.10}$$

with $m = 2\sqrt{\beta}$. By direct substitution of the Bäcklund equations a simplified expression of the bulk energy is acquired

$$\mathcal{H}_{\text{bulk}} = \int dx (2m^2 uu^*) . \tag{4.11}$$

When the above expression is integrated throughout space, the result can be identified with the mass of the soliton solution u. However, now the integration is over the half line and specifically over the $[-\infty, 0]$ region.

The same equations can be used to express the $\mathcal{H}_B$ term of (4.9) . Specifically, the boundary constant C is determined by direct comparison of the Bäcklund equations of (2.22) and the boundary condition which appears in (3.24)

$$C = \frac{m}{2} \left(\delta \, e^{i(\theta+\Omega)} + \frac{1}{\delta} \, e^{-i(\theta+\Omega)} \right) . \tag{4.12}$$

with θ given by (2.24). At $x = 0$ and assuming that $V = 0$, the above expression simplifies to

$$C = \pm \frac{m}{\sqrt{1 + \tan^2(a) \coth^2(m \cos(a) x_0)}} . \tag{4.13}$$

This implies that $|C| \leq |m|$. Since both m and C are defined by the theory, the above relation is true only for specific choices of the boundary parameter C. Alternatively, one can think of this restriction emerging from the fact that for $|C| > |m|$, no choice of x_0 satisfies the boundary condition.

In the case where we choose $u = 0$ as a possible vacuum, the only remaining term in the total energy is

$$\mathcal{H}_{\text{tot}} = -2C . \tag{4.14}$$

Alternatively, one can consider a one-soliton solution where V is set to zero which appears in (2.49). In this case, both terms of (4.8) depend on the initial position of the soliton. However, after some calculations, the x_0 dependence drops out and the total energy is given by a the following expression

$$\mathcal{H}_{\text{tot}} = 2m \cos(a) . \tag{4.15}$$

It is far from obvious, which vacuum choice provides the minimum energy configuration. To determine this, one has to look at the expression of the boundary constant C in (4.13). This can be rewritten in the following form

$$y^2 + \frac{y^2 - y^4}{y^2 + F^2} = \cos^2(a) , \tag{4.16}$$

where

$$F^2 = \sinh^2(m\cos(a)x_0) \ , \quad y = \frac{C}{m} \ . \tag{4.17}$$

In the above relation m and C should be treated as fixed parameters, while F can be varied through x_0. The left hand side of (4.16) is monotonically decreasing as F increases since $0 < y^2 < 1$. We observe the following

$$\cos^2(a) = 1 \text{ when } \quad F \to 0 \ ,$$
$$\cos^2(a) = y^2 \text{ when } \quad F \to \pm\infty \ . \tag{4.18}$$

This shows that moving the soliton away from the boundary decreases the energy of the system. On the extreme case where the soliton is placed at infinity, the model behaves as if no soliton exists, and the only contribution is the boundary term which coincides with the vacuum solution of $u = 0$. On the contrary as the soliton is placed closer to the boundary the energy increases. The maximum energy occurs when $F = 0$ at which point $\cos^2(a) = 1$ so that $H_{\text{tot}} = 2m$ which is greater than the energy $H_{\text{tot}} = 2C$ of the $u = 0$ vacuum.

Although the choice of vacuum in the bulk seems to be the most suitable choice in the boundary case too, one cannot rule out multi-soliton solutions that might provide lower energy configurations. This demands tedious calculations and remains as one of the open questions for this model.

4.2 Soliton reflections

In this section we investigate the reflection of solitons from the boundary. Mathematically this can be represented by a two-soliton solution satisfying the boundary condition. One of the solitons represents the incoming soliton whilst the other represents the reflected one. The point where the two solitons actually meet along the whole line as well as the phase shift due to their collision create an overall time-delay effect which can be calculated directly through the parameters of the scattering. This time-delay can be attributed to the interaction of the soliton with the boundary.

However, the most difficult step is to determine the restrictions that have to be imposed so that the two-soliton solution satisfies the boundary condition

$$\partial_1 u_{2s} = -C u_{2s}\sqrt{1 - u_{2s}u_{2s}^*} \ . \tag{4.19}$$

Energy and charge conservation laws demand that both the mass and the charge of the soliton are conserved by the boundary. This restricts the choice of the charge parameters a_1, a_2 to be either equal or opposite.

Due to the large expressions involved in the calculation, one is forced to expand both sides of the equation (4.19) to a Taylor series in exponentials of t, and match each term of the same order. Each term provides us with an equation involving the boundary parameter C. As mentioned in the previous section, the boundary constant has to be a real parameter. The real and imaginary parts of the equation yield two constrains on the parameters.

Let us consider this in more detail. We begin with a two-soliton solution, where the parameters are chosen in such a way so as to describe a soliton-soliton scattering. In
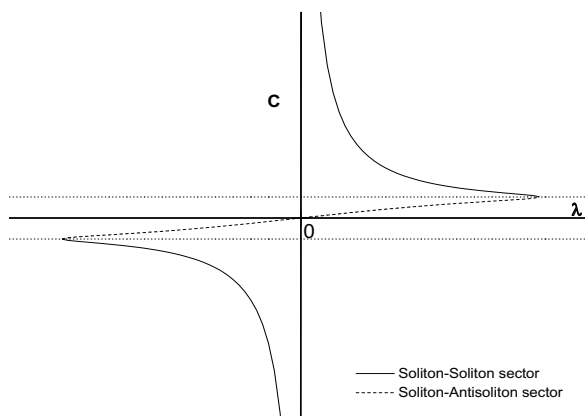
Figure 2: C in terms of λ.

this case, the charge parameters are taken to be opposite $a_1 = -a_2$ and the Bäcklund parameters to be $\delta_1 = -1/\delta_2$.

Furthermore, we adopt the following parametrisation which is more natural

$$\frac{K_1}{K_2} = e^\lambda \ , \quad \frac{J_1}{J_2} = e^{i\zeta} \ , \quad V = \tanh(\vartheta) \ . \tag{4.20}$$

After both sides of the boundary equation are expanded as a Taylor series in time, we can discard the imaginary parts from all terms by using the following relation

$$\sin(\zeta) = -\tanh(\vartheta)\tan(a)\sinh(\lambda). \tag{4.21}$$

When the above equation is used the infinite set of equations collapse to a single constraint

$$C = m\cos(a)\cosh(\vartheta) \ \left(\frac{\cos(\zeta) + \cosh(\lambda)}{\sinh(\lambda)}\right) \ . \tag{4.22}$$

When the shift parameters are fixed according to the above relations, the two-soliton solution satisfies the boundary condition and this process describes a soliton being reflected by the boundary.

The fact that only relative shifts in both normal and internal U(1) space are important should be expected from time translational and U(1) invariance of the model. The non-topological solitons in the CSG theory are reflected as solitons carrying the same charge Q. This is because the boundary potential does not break the U(1) symmetry since it depends only on $|u|$.

We shall consider a and ϑ as fixed parameters and use equations (4.21) and (4.22) to determine the parameter λ in terms of C. By eliminating the parameter ζ, one ends up
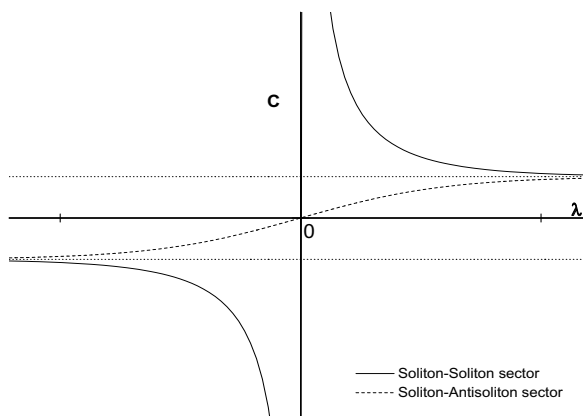
Figure 3: C in terms of λ for chargeless case.

with the following quadratic equation for C

$$C^2 - 2\,\frac{Cm\cos(a)\cosh(\vartheta)\cosh(\lambda)}{\sinh(\lambda)} + \left(\sinh(\vartheta)^2 + \cos(a)^2\right)m^2 = 0 \ \ . \qquad (4.23)$$

The solutions of the above equation can be plotted to present the dependence on λ. The plot involves two branches (figure 2) due to the sign ambiguity, which are mutually exclusive. The plot shows that a soliton can always be reflected by the boundary. The branches meet at the points

$$C = \pm m\sqrt{\cos^2(a) + \sinh^2(\vartheta)} \ , \ \ \coth(\lambda) = \frac{\sqrt{\sinh^2(\vartheta) + \cos^2(a)}}{\cos(a)\cosh(\vartheta)} \ \ . \qquad (4.24)$$

In the limit $a \to 0$ the two branches of the plot can be identified with the soliton-soliton and soliton-antisoliton sector of the reflection process at the sine-Gordon limit (figure 3). For fixed values of ϑ and $a = 0$, it is the value of the boundary constant C which determines whether a soliton is reflected as a soliton or an antisoliton. For C small, a soliton is reflected as an antisoliton (Neumann boundary conditions for $C = 0$), while for C large a soliton is reflected as a soliton (Dirichlet boundary conditions for $C = \infty$). For $C = m\cosh(\vartheta)$ the branches do not meet as in the CSG case. This specific value of C corresponds to a logarithmic divergence that appears in the classic time delay for the sine-Gordon case. These results coincide with the results derived by previous treatments of the boundary sine-Gordon model [2].

4.3 The classical time delay

The time delay which appears at the scattering of a soliton off the boundary, can be calculated directly from the asymptotic values of the solution at $t = \pm\infty$. We begin with

the two-soliton solution and change to a frame of reference which moves with the incoming soliton (i.e. $x = Vt + \tilde{x}$. In the limit $t = -\infty$ the solution becomes

$$S_- = \lim_{t \to -\infty} u_{2s} = \frac{\cos(a) \ e^{i(A_1+B_1)}}{\cosh(P(-\tilde{x}+x_1)+r)} \ , \qquad (4.25)$$

where

$$A_1 = \frac{P\sin(a)}{\cos(a)} \left[(1-V^2)t - V\tilde{x} - y_1\right] \quad , \quad \tan(B_1) = -\frac{V\sin(a)}{\cos a} \qquad (4.26)$$

and

$$P = \frac{m\cos(a)}{\sqrt{1-V^2}} \quad , \quad r = \frac{1}{2}\ln\frac{V^2}{\cos^2(a)+V^2\sin^2(a)} \quad . \qquad (4.27)$$

The parameters x_i and y_i represent regular shifts that were introduced in (2.42). The solution, as expected, describes a single incoming soliton at early time far away from the boundary.

We repeat the same calculation, but now we change to the frame of reference of the outgoing soliton (i.e. $x = -Vt$) and calculate the limit of the two-soliton solution at $t = +\infty$ which yields

$$S_+ = \lim_{t=+\infty} u_{2s} = \frac{\cos(a) \ e^{i(A_2+B_2+\pi)}}{\cosh(P(-\tilde{x}-x_2)+q)} \ , \qquad (4.28)$$

where

$$A_2 = \frac{P\sin(a)}{\cos(a)} \left[(1-V^2)t + V\tilde{x} - y_2\right] \quad , \quad \tan(B_2) = \frac{V\sin(a)}{\cos(a)} \ , \qquad (4.29)$$

and

$$q = \frac{1}{2}\ln\frac{V^2}{\cos^2(a)+V^2\sin^2(a)} \quad . \qquad (4.30)$$

Once again this is a single soliton solution representing the reflected soliton far away from the boundary wall.

The asymptotic solutions S_+, S_- contain all the information needed to calculate the time-delay. The latter is a combination of two separate events. Firstly, a phase shift is induced during the scattering of the two solitons. Before the two solitons re-emerge as two separate entities, the reconfiguration of the solution creates a phase shift which is equivalent to a time delay. Secondly, the centre of mass of the two-soliton solution does not necessarily lies at the boundary. This implies that the two solitons actually meet at a different point than $x = 0$. This creates again a time delay which may be either positive or negative corresponding to an attractive or repulsive boundary potential respectively.

Ignoring any interaction between the two solitons, one can project the trajectories of S_+ and S_- on $x - t$ diagram and find the point where these cross (figure 4). The distance of this point from the boundary is proportional to the time delay $\Delta\tau$ which in the diagram is given by the distance (AC). The time delay corresponds to the time interval in which the soliton appears to be absorbed by the boundary before it reemerges as a well defined entity . The two solitons move across the following lines

$$S_- : \quad t = \frac{1}{V}\left(P(-x+x_1)+r\right) \ ,$$

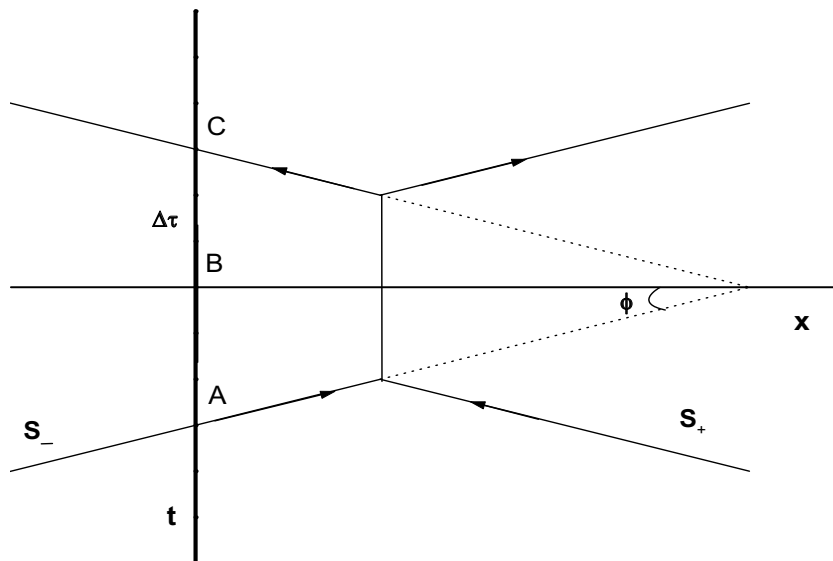$$S_+ : \quad t = \frac{1}{V}\left(P(x+x_2)-q\right) \ ,$$

Figure 4: Scattering diagram.

as dictated by (4.28) and (4.25). The lines cross at

$$
x_0 = \frac{1}{2}\left(x_1 - x_2 + \frac{r+q}{P}\right) = \frac{x_1 - x_2}{2} + \frac{r}{P} \ , \tag{4.31}
$$

since $r = q$. The time delay is finally

$$
\Delta\tau_{\mathrm{CSG}} = \frac{2x_0}{V} = \frac{(x_1 - x_2)}{V} + \frac{\sqrt{1-V^2}}{mV\cos(a)}\ln\left(\frac{V^2}{\cos^2(a) + V^2\sin^2(a)}\right) \ . \tag{4.32}
$$

In the expression above, the first term of the right-hand side represents the time delay caused by the non-symmetric character of the solution with respect to the boundary. In the special case where $x_1 = x_2$, the centre of mass lies on the boundary and the term vanishes. The second term is independent of the initial position of the two solitons or the boundary potential and is caused by the phase shift of the scattering process.

The relative position of the two solitons are however fixed according to the constraint equations (4.21) and (4.22) which ensure that solution satisfies the boundary condition. Specifically the parameter λ corresponds exactly to the $x_1 - x_2$ difference up to the overall factor P. It is thus possible to express the time delay in terms of the boundary constant, by solving the constraint equations and substituting the relative position of the solitons. We choose to express the velocity parameter V in terms of the rapidity ϑ for simplicity reasons. After a few straightforward calculations we recover the following expression for the time delay

$$
\Delta\tau_{\mathrm{CSG}} = \frac{\ln Q}{2m\cos(a)\sinh(\vartheta)} \tag{4.33}
$$

where

$$Q = \frac{\sinh^4 \vartheta \left((\cos^2(a) + \sinh^2(\vartheta))m^2 + 2Cm\cos(a)\cosh(\vartheta) + C^2\right)}{(\cos(a)^2 + \sinh^2(\vartheta))^2 \left((\cos^2(a) + \sinh^2(\vartheta))m^2 - 2Cm\cos(a)\cosh(\vartheta) + C^2\right)} \quad (4.34)$$

In the special limit of $a = 0$, the time delay for the sine-Gordon model is recovered

$$\Delta\tau_{\mathrm{SG}} = \frac{1}{m\sinh(\vartheta)} \ln\left(\tanh^2(\vartheta)\frac{m\cosh(\vartheta) + C}{m\cosh(\vartheta) - C}\right) \quad . \quad (4.35)$$

This is exactly the time delay calculated for the sine-Gordon theory in the presence of a boundary [2] for the restricted class of boundary conditions which admit $\phi = 0$ as a vacuum to which the chargeless limit of CSG correspond.

4.4 Boundary bound states

In this section we examine the spectrum of bound states. Once again, for the the boundary condition to be satisfied we need to restrict some of the parameters in the solution.

The simplest bound state that we can have is the static single soliton that was introduced in (2.49). The solution is not really static, as the imaginary phase survives the setting of the speed parameter V to zero. The solution is static only in the sense that the centre of mass doesn't translate in the x direction, although the wave oscillates with fixed angular velocity $\omega = m\sin(a)$.

When a boundary is introduced a static soliton can satisfy the boundary condition for $|C| \leq |m|$ when its position is fixed according to equation (4.13). At the chargeless limit any time dependence vanishes and the solution collapses to a static single soliton of the sine-Gordon theory, fixed at the boundary.

Breathers that have been constructed by the method described in section (2.5) can also be shown to satisfy the boundary condition. The condition that C is real still holds. However, all the arbitrary phase shifts are now real numbers and constrained. We examine breather solutions that emerge from the soliton-soliton case. Just as before, the solution does satisfy the boundary condition with some restrictions involving the arbitrary parameters. Once more a Taylor expansion of the boundary equation is needed. The parametrization used in this case is

$$K_1 = e^\lambda \ , \quad J_1 = e^\zeta \ , \quad V = \tan(\vartheta) \ , \quad (4.36)$$

while the parameters K_2 and J_2 have been been properly fixed so that this is a breather solution. The first restriction needed for the solution to satisfy the boundary condition is

$$\sinh(2\zeta) = -\tan(a)\tan(\vartheta)\sinh(2\lambda) \quad . \quad (4.37)$$

The parameter ϑ plays the role of the rapidity, which has now been analytically continued. The second restriction which completes the necessary requirements for a boundary bound state is

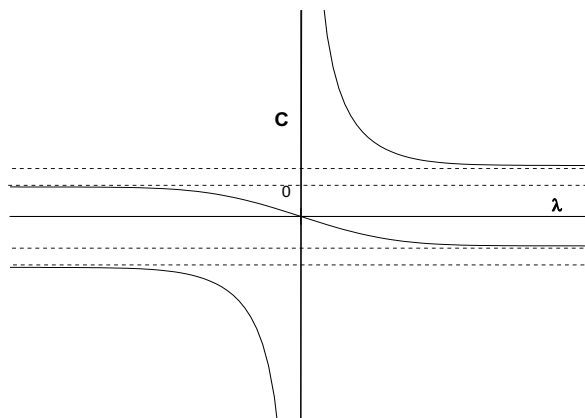$$C = m\cos(a)\cos(\vartheta)\left(\frac{\cosh(2\zeta) - \cosh(2\lambda)}{\sinh(2\lambda)}\right). \quad (4.38)$$

Figure 5: C in terms of λ for Breather case.

Both relations can be recovered by analytical continuation of the corresponding relations of (4.21) and (4.22) after the necessary restrictions for a breather solution have been already taken into account. It is instructive to examine the relation between the parameters C and λ (figure 5) since it provides valuable insight to the structure of bound states. There are two distinct regions of values of C that do not correspond to any bound state. This regions are defined by the limit values

$$C = \pm m \left(\cos(a) \cosh(\theta) \pm \sin(a) \sinh(\theta) \right) \quad . \tag{4.39}$$

At the chargeless limit, the regions collapse to the single values

$$C = \pm m \cosh \theta \ , \tag{4.40}$$

which coincided with the logarithmic divergence appearing in the time delay for the soliton reflection.

4.5 Particle reflections

In this section we consider the spectrum of particles and their reflection factors in the presence of a boundary.

For small fluctuations around the vacuum $u = 0$, the boundary condition (3.24) becomes

$$\partial_1 \epsilon(x,t) = -C \epsilon(x,t). \tag{4.41}$$

In order to calculate the reflection factor when particles bounce of the boundary wall, we substitute in the last relation the particle solutions presented in (2.18). The constant A of the right propagating waves is taken to be one, since it has to do with the characteristics of

the particle beam. The reflection factor is identified with the constant B, which corresponds to a phase change as the particles encounter the boundary

$$B = \frac{ik + C}{ik - C} \; .\tag{4.42}$$

The reflection factor, as expected, depends on C which as stated before, appears as a free, real parameter in the boundary condition. For $C = 0$, the reflection factor is equal to $B = 1$ and no phase appears between the two waves upon their scattering off the boundary. This is consistent with the fact that the boundary term is proportional to the boundary constant, so when C is set to zero, the boundary term vanishes.

Particle solutions can be related to bound states through the pole appearing in B. Indeed one may choose $k = -iC$ and apply this to a solution of the form $\frac{1}{B}\epsilon(x,t)$. The remaining terms depend explicitly on the boundary constant

$$\epsilon(x,t) = e^{-i\left(\sqrt{m^2 - C^2}t + -iCx\right)} \; .\tag{4.43}$$

This solution is square integrable only for a specific range of values for the boundary constant. Specifically if C is positive then the solution is not square integrable since it is exponentially increasing as $x \to \infty$.

When $-m < C < 0$, then $\epsilon(x,t)$ represents a square integrable exponentially decreasing solution as $x \to \infty$. It oscillates with constant angular velocity $\omega = \sqrt{m^2 - C^2}$ and is therefore a stable bound state. It can also be viewed as the tail of a static one-soliton solution satisfying the boundary condition, with the parameters adjusted in such a way its centre of mass goes to positive infinity. Examining the condition (4.13) for the static soliton to obey the boundary condition, this limit can be achieved as $x_0 \to \infty$, i.e. we must take the charge is such a way that $C = -m\cos a$.

Finally in the region $C < -m$, the solution can increase exponentially in time. This shows that the vacuum solution $u = 0$ is no longer stable. In fact the particle behaviour which corresponds to a small perturbation around the vacuum seems to be ever increasing. This instability can be understood through a rather impressive mechanism in which a chargeless soliton is emitted from the boundary, effectively changing the value of C so that $u = 0$ is now stable.

Recall from section (2.4) that for a chargeless soliton we should take the opposite sign for $\sqrt{1 - uu^*}$ on each side of the centre of the soliton where $|u| = 1$. The instability can be viewed as a left moving chargeless soliton which approaches the boundary from $x = \infty$. In the beginning while the soliton is far away from the boundary $u = 0$ so that the boundary potential of (4.9) is $H_B = -2C$. As the centre of the soliton passes through $x = 0$, the sign of the square root in the boundary potential changes. As the soliton moves to $x = -\infty$, u returns to 0 near the boundary but now we take the boundary energy with the opposite sign $H_B = 2C$. Effectively the sign of C has been flipped to a positive value. The energy released from the boundary is $4C > 4m$, which is greater than the rest mass of a single chargeless soliton. At $C = -m$, the soliton is emitted with infinitesimal velocity. As C decreases, more energy is given up by the boundary and the soliton can be emitted with larger V. This process agrees with the infinite time-delay effect which was encountered in

the soliton reflections section. The soliton emission represents the time reversal picture of that effect in the chargeless soliton case (figure 3). It follows that we need never consider the situation where $C < -m$.

5. Discussion

The CSG model is one of the simplest generalisations of the sine-Gordon theory, but nonetheless has a rich and fascinating mathematical structure.

In the first part of the paper we have examined the spectrum of the theory in the bulk and written down explicit two-soliton solutions within the framework of the matrix potential. We also demonstrated how to construct breather solutions in an elegant way avoiding the problems that arise by the analytical continuation of the parameter V. There are two ways in which soliton solutions of the CSG model differ qualitatively from those of the Sine-Gordon model. Firstly, the solitons can be charged, so that whilst there centre of mass remains fixed, the solutions are not static. The second feature is that the CSG does not possess degenerate vacua, and so the solitons are non-topological. There is therefore no distinction between solitons and antisolitons which can be interchanged by a continuous variation of the charge parameter a. Nevertheless, the topological nature of the sine-Gordon theory can be recovered as the choice of branch cut of $\sqrt{1 - |u|^2}$ in the chargeless limit as was demonstrated in section 2.4 . A direct consequence of the non-topological nature of the CSG soliton is that the breather solution can collapse to a single soliton when the parameter V is properly fixed, tying in with the picture presented in [18] that particle and solitons can be identified in the quantum limit.

In the second part of the paper we introduced a boundary term in the CSG Lagrangian and demanded that the system remains integrable. First we constructed low-spin conserved quantities of the theory using abelianisation of the Lax pair, and then derived suitable boundary conditions in order to preserve these. In the presence of a boundary, we examined the vacuum structure and showed that the bulk vacuum $u = 0$ remained the true vacuum in the boundary case.

Soliton reflections off the boundary were also studied and the necessary constraint equations were written down in terms of the phase shift parameters. The set of equations was derived by demanding that the two-soliton solution satisfies the boundary condition. Moreover the time delay induced by the scattering process was calculated in terms of the boundary constant C and was found to coincide in the chargeless limit with the time delay of the sine-Gordon theory.

Finally we looked for classical solutions corresponding to boundary bound states. We found that it was possible to construct both bound soliton and bound breather solutions. We also found a bound state in the particle spectrum. This was unstable when parameter C associated with the boundary energy was in the range $C < -m$. In this case the boundary emits a chargeless soliton, effectively changing the sign of the boundary parameter C to $C > m$.

We end our discussion by pointing out a few aspects of the model that appear quite interesting and deserve further study. As mentioned in the introduction the CSG theory

was used to generalise the existing field theory approach of optical pulse propagating in a non-linear medium. A physical interpretation of the results appearing in this paper would be extremely interesting, especially the physical meaning of breather solutions and their application to physical geometries.

An obvious extension of our results is to consider the quantum case of the boundary CSG model. The S-matrix for the model in the bulk, which corresponds to perturbed Z_n parafermions, is known. It would be interesting to see if one can find a quantum reflection matrix, compatible with this S-matrix and with the classical results presented in this paper. As the simplest case in the family of homogeneous sine-Gordon theories, the results might shed light on the more complicated models in the family.

Acknowledgments

A. Detailed derivation of the boundary condition

In order to construct an infinite number of conserved quantities when a boundary is introduced, we need to express the parity odd terms of the equations of motion as total time derivatives. The corresponding term of order λ^2 is in terms of the fields u and u^*

$$
-2 \frac{(\partial_1 u^*) \left(\partial_0^2 u + \partial_1^2 u \right)}{1 - uu^*} - 4 \frac{(\partial_0 u^*) \partial_1 \partial_0 u}{1 - uu^*} + 2 \frac{(\partial_1 u) \left(\partial_0^2 u^* + \partial_1^2 u^* \right)}{1 - uu^*} + 4 \frac{(\partial_0 u) \partial_1 \partial_0 u^*}{1 - uu^*}
$$

$$
+2 \frac{u^* (\partial_1 u)^2 \partial_1 u^*}{(1 - uu^*)^2} - 2 \frac{u (\partial_1 u^*)^2 \partial_1 u}{(1 - uu^*)^2} - 4 \frac{u (\partial_0 u^*)(\partial_1 u^*) \partial_0 u}{(1 - uu^*)^2} + 2 \frac{u^* (\partial_0 u)^2 \partial_1 u^*}{(1 - uu^*)^2}
$$

$$
-2 \frac{u (\partial_0 u^*)^2 \partial_1 u}{(1 - uu^*)^2} + 4 \frac{u^* (\partial_0 u)(\partial_1 u) \partial_0 u^*}{(1 - uu^*)^2} + 4 \left(u \partial_1 u^* - u^* \partial_1 u \right) \beta \quad .
$$

Since we need the above expression to be a total time derivative we can eliminate any second order spatial derivatives of the fields by using the equations of motion of (2.1)

$$
4 \frac{\partial_0 u \partial_0 \partial_1 u^*}{1 - uu^*} - 4 \frac{\partial_0 u^* \partial_0 \partial_1 u}{1 - uu^*} - 4 \frac{(\partial_1 u \partial_1 u^* + \partial_0 u \partial_0 u^*)(u \partial_1 u^* - u^* \partial_1 u)}{(1 - uu^*)^2}
$$

$$
-4 \beta(u \partial_1 u^* - u^* \partial_1 u) + 4 \frac{\partial_1 u \partial_0^2 u^*}{1 - uu^*} - 4 \frac{\partial_1 u^* \partial_0^2 u}{1 - uu^*} \quad , \tag{A.1}
$$

We take advantage of the fact that we are free to add total time derivatives on this expression, since this represents a conserved quantity. The expression simplifies significantly by adding the following term

$$
\partial_0 \left(4 \frac{\partial_1 u^* \partial_0 u - \partial_1 u \partial_0 u^*}{1 - uu^*} \right) \quad , \tag{A.2}
$$

which yields

$$
-4 \frac{(\partial_1 u^*) \partial_0^2 u}{1 - uu^*} + 4 \frac{(\partial_1 u) \partial_0^2 u^*}{1 - uu^*} + 4 \frac{(\partial_0 u) \partial_1 \partial_0 u^*}{1 - uu^*} - 4 \frac{(\partial_0 u^*) \partial_1 \partial_0 u}{1 - uu^*}
$$

$$+4\,\frac{\left(-u\partial_1\,u^*+u^*\,\partial_1\,u\right)\left(\left(\partial_1\,u^*\right)\partial_1\,u+\left(\partial_0 u\right)\partial_0 u^*\right)}{\left(1-uu^*\right)^2}$$
$$+4\,\left(-u\partial_1\,u^*+u^*\,\partial_1\,u\right)\beta\quad. \tag{A.3}$$

We are looking for boundary conditions that are of the form

$$\partial_1 u = F(u,u^*)\quad,\quad \partial_1 u^* = G(u,u^*)\quad, \tag{A.4}$$

where F and G are functions of the fields not involving derivatives. By direct substitution of the above into (A.3) we get

$$4\,\frac{\left(2\frac{\partial G}{\partial u}(1-uu^*)+Gu^*\right)(\partial_0 u)^2}{(1-uu^*)^2}-4\,\frac{(\partial_0 u^*)^2\left(2\frac{\partial F}{\partial u^*}(1-uu^*)+Fu\right)}{(1-uu^*)^2}$$
$$8\,\frac{(\partial_0 u^*)\left(-\frac{\partial F}{\partial u}+\frac{\partial G}{\partial u^*}\right)\partial_0 u}{1-uu^*}+4\,\frac{(-uG+u^*F)\,GF}{(1-uu^*)^2}$$
$$+4\,(-uG+u^*F)\,\beta\quad.$$

The expression above does represent a total derivative when all terms are forced to vanish by selecting suitable functions F and G. The two separate differential equations that appear involving the undefined functions

$$2(1-uu^*)\frac{\partial F}{\partial u^*}+uF=0\quad,$$
$$2(1-uu^*)\frac{\partial G}{\partial u}+u^*G=0\quad,$$

can easily be solved to yield

$$F(u,u^*)=S_1(u)\sqrt{1-uu^*}\quad,\quad G(u,u^*)=S_2(u^*)\sqrt{1-uu^*}\quad. \tag{A.5}$$

In addition, the last two terms in (A.5) imply that

$$F=\frac{u}{u^*}\,G\quad, \tag{A.6}$$

Using the above relation and solutions of (A.5) into the remaining terms of (A.5), we can determine the remaining undefined functions S_1 and S_2. The final form of the boundary conditions are

$$\partial_1 u = -Cu\sqrt{1-uu^*}\,,$$

$$\tag{A.7}$$

$$\partial_1 u^* = -Cu^*\sqrt{1-uu^*}\quad.$$

where C is a real constant.

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